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Numerical Study of Integral-Algebraic Equations of Index-3 by Piecewise Polynomial Multi-Step Collocation Method

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Abstract

Integral-algebraic equations (IAEs) of index-3 are known as high index IAEs. Similar to differential-algebraic equations (DAEs), the index of a IAE is a measure of its complexity, with higher indices indicating more challenging systems (a measure of the system's complexity and numerical challenges). In this paper, we apply multi-step collocation method to solve IAEs of index-3. We combine the ideas of multi-step methods and collocation methods where the solution is approximated by matching conditions at specific points. Convergence analysis is studied and it examines whether and how quickly a sequence of approximations generated by the multi-step collocation method approaches the desired solution. Next, for comparison, we use a one-step collocation technique to numerically solve the equation, showcasing the effectiveness and precision of the multi-step collocation approach.

Keywords: high-index integral-algebraic equations; polynomial multi-step collocation methods; convergence analysis.

1 Introduction

Differential-algebraic equations (DAEs) [3] are systems that combine differential equations with algebraic constraints. The index of a DAE is a measure of its complexity, with higher indices indicating more challenging systems. The most general form of a differential-algebraic equations [20] is :

$$\mathbf{F}\left(t, \mathbf{y}, \mathbf{y}'\right) = \mathbf{0}, \tag{1}$$

where $t \in \mathbb{R}$ is the independent variable, $\mathbf{y}(t) \in \mathbb{R}^d$ is the vector of unknown state variables, and $\mathbf{y}'$ is its derivative. System (1) is said to be a system of differential-algebraic equations if the Jacobian $\frac{\partial \mathbf{F}}{\partial \mathbf{y}'}$ is singular for all values of its arguments. The general semi-linear version of the DAE (1) is given by

$$\mathbf{E}(t)\mathbf{y}' + \mathbf{b}\left(t, \mathbf{y}\right) = \mathbf{0}, \tag{2}$$

where $\mathbf{E}(t)$ is continuous and singular matrix. The semi-explicit form of the linear DAE

$$\mathbf{E}(t)\mathbf{y}' + \mathbf{K}(t)\mathbf{y} = \mathbf{q}(t), \tag{3}$$

where $\mathbf{E}(t) = diag\left(\mathbf{I}_{d_1}, \mathbf{0}_{d_2}\right)$, $\mathbf{I}_{n_1}$ is $d_1 \times d_1$ identity matrix and $d_1 + d_2 = d$.

The differentiation index of a DAE [15] is the minimum number of times that all or part of the system must be differentiated with respect to the independent variable t in order to transform the DAE into an explicit system of ordinary differential equations (ODEs), known as the underlying ODE. The semi-explicit index-3 DAE system [16] has the form:

$$x' = F(t, x, y, u), \tag{4}$$

$$y' = G(t, x, y), \tag{5}$$

$$0 = H(t, y), \tag{6}$$

where $H_y G_x F_u$ has a bounded inverse. The system is index-3 because three differentiations of the constraint (6) are required to transform it into an ODE system. Also, for index-3 DAEs the algebraic constraints directly affect the highest derivatives, typically more stiff than lower-index DAEs and numerical solutions [10] may drift from the constraint manifold. These kind of systems arise in multi-body dynamics with holonomic constraints, mechanical systems with position-level constraints and electrical networks with topological constraints. In [1], reproducing kernel algorithm was studied for handling DAEs and in [2], RKHS method was considered to investigate the numerical solution of integro-differential algebraic systems.

Integral-algebraic equations (IAEs) combine second kind integral equations with first kind integral equations. The index of an IAE system indicates how many times the equations must be differentiated to transform them into a system of integral equations of the second kind. Here, we consider the semi-explicit index-3 IAEs in the following form:

$$\begin{cases} u_1(x) = f_1(x) + \displaystyle\int_0^x H_{11}(x,y)u_1(y)dy + \int_0^x H_{12}(x,y)u_2(y)dy \\ \qquad\qquad + \displaystyle\int_0^x H_{13}(x,y)u_3(y)dy, \\ u_2(x) = f_2(x) + \displaystyle\int_0^x H_{21}(x,y)u_1(y)dy + \int_0^x H_{22}(x,y)u_2(y)dy, \\ \quad 0 = f_3(x) + \displaystyle\int_0^x H_{32}(x,y)u_2(y)dy, \qquad x \in J = [0, X], \end{cases} \tag{7}$$

where $f_i, H_{ij}(i,j = 1,2,3)$ are the given functions. Based on differential index, the index is a measure of how far an IAE system is from being a system of second kind Volterra integral equations (VIEs). Intuitively, it tells you how many times you need to differentiate the entire system to convert it into a system of second kind VIEs.

1. Index-0: Already a system of second kind VIEs.
2. Index-1: Requires one differentiation. Numerical methods are generally stable and well-behaved.
3. Higher Index (2, 3,...): Requires multiple differentiations. This is where severe numerical difficulties arise.

Due to significant mathematical and computational challenges, the majority of research on the numerical analysis of IAEs has historically focused on problems with an index of three or less, a category that encompasses many foundational applications like constrained mechanical systems.

DAEs and IAEs provide a framework for formulating various applied mathematics challenges, including inverse Stefan problems [11], identification of memory kernels [19], heat-transfer [25], rigid-body dynamics [9], network analysis [12] and control systems [5]. In reference [23], the researcher investigated approximate solutions for IAEs (7) using one-step collocation techniques. Global convergence analysis was investigated and optimal order of convergence was obtained. In [24], the authors examined unique solution for IAE systems considering a novel index definition and employed piecewise continuous collocation methods for numerical resolution. Block pulse functions have been studied to find numerical solution of IAEs in [4]. In [6], the authors investigated sufficient conditions for the existence and uniqueness of solution to systems of two-dimensional IAEs. Based on Adams's quadrature and extrapolation formulas, construction of the numerical methods for the class of linear IAEs has been studied in [7]. The tractability index for linear IAE systems has been defined in [21] considering the ill-posed nature of first-kind VIEs. A new definition based on the left index was defined for IAEs of the Hessenberg type in [24]. Convergence analysis of the multistep collocation method was investigated for IAEs of index-1 in [27]. Convergence of one-step methods for Volterra integral equations of the second kind was investigated in [14]. In [26] the authors studied multi-step collocation approximations to solutions of first-kind Volterra integral equations. Direct multi-step method has been proposed in [18] to solve retarded and neutral delay differential equations. Also, in [17], the implicit multi-step method was formulated to solve the initial-value problems of neutral delay Volterra integro differential equations.

Multi-step methods use information from several previous steps to compute the next value. They are efficient but can be complex to derive for high orders. On the other hand, collocation methods find a polynomial that satisfies the integral equation exactly at a set of collocation points within a single step. A polynomial multi-step collocation method takes a multi-step view by considering a time interval that spans several steps, but then uses the collocation principle to determine the solution across that entire interval. Polynomial multi-step collocation methods are advanced numerical methods for solving Volterra integral equations. These methods combine ideas from collocation and multi-step methods to achieve higher-order accuracy.

The paper is organized as follows:
Section 2 examines the existence and uniqueness conditions for solutions of index-3 IAEs and introduces a polynomial multi-step collocation method aimed at discretizing (7), utilizing widely recognized interpolation polynomials. Subsequently, in Section 3, we develop and analyze the

convergence of numerical solutions. This is succeeded by the examination of three test problems in Section 4 to validate theoretical findings.

2 Approximation

2.1 Unique solution of index-3 IAEs

The well-posedness of the index-3 IAEs (7) ensuring a solution exists and is unique, depends on the solutions of index-2 IAEs can be studied by the following Theorem:

Theorem 2.1. *Let $m \geq 0$ and*

1. $H_{1l} \in C^{m+1}(O)$ *for* $l = 1, 2, 3$ *and* $O = \{(x, y) : 0 \leq y \leq x \leq X\}$,
2. $H_{2l} \in C^{m+2}(O)$ *for* $l = 1, 2$,
3. $H_{32} \in C^{m+3}(O)$ *and* $|H_{32}(x, x)H_{21}(x, x)H_{13}(x, x))| \geq h_0 > 0, \forall x \in J$,
4. $f_1 \in C^{m+1}(J), \ \ f_2 \in C^{m+2}(J), \ \ f_3 \in C^{m+3}(J)$ *and* $f_3(0) = 0$,

then the system of IAEs (7) admits a unique solution $u_1, u_2, u_3 \in C^m(J)$.

Proof. By differentiating the third equation of (7) with respect to x and replacing u_2 using the second equation, we obtain index-2 IAEs as defined in [13]

$$\begin{cases} u_1(x) = f_1(x) + \displaystyle\int_0^x H_{11}(x, y)u_1(y)dy + \int_0^x H_{12}(x, y)u_2(y)dy \\ \qquad\qquad + \displaystyle\int_0^x H_{13}(x, y)u_3(y)dy, \\ u_2(x) = f_2(x) + \displaystyle\int_0^x H_{21}(x, y)u_1(y)dy + \int_0^x H_{22}(x, y)u_2(y)dy, \\ 0 = \bar{f}_3(x) + \displaystyle\int_0^x H_{32}(x, x)H_{21}(x, y)u_1(y)dy + \int_0^x \bar{H}_{32}(x, y)u_2(y)dy, \end{cases} \tag{8}$$

where $\bar{f}_3(x) = f_3'(x) + H_{32}(x, x)f_2(x)$, $\bar{H}_{32}(x, y) = H_{32}(x, x)H_{22}(x, y) + \frac{\partial H_{32}(x,y)}{\partial x}$.

Under suitable regularity conditions, this problem becomes equivalent to index-2 IAEs, enabling us to establish the unique solution. The proof is concluded by application of Theorem 4.2 from [22]. □

2.2 Multi-step collocation method

To develop a multi-step collocation scheme for equation (7), we begin by assuming the following framework:

$$J_h := \{x_n = nh, \quad n = 0, 1, \ldots, N \quad (x_N = X, h = X/N, \ N > 0)\}.$$

Let the partition P_h be defined as

$$P_h := \{x_{n,i} = x_n + s_i h : 0 < s_1 < \cdots < s_m \leq 1 \; (0 \leq n \leq N-1)\},$$

such that $\{x_{n,i}\}$ and $\{s_i\}$ represent collocation points and parameters, respectively. On each subinterval $\epsilon_n = [x_n, x_{n+1}]$, for $t \in [0,1], n \geq r-1$, the $r+1$-step collocation solution (by introducing in the collocation polynomial the dependence on r previous time steps) is expressed as:

$$w_1(x_n + th) = \sum_{i=0}^{r-1} \Omega_i(t) u_{1,n-i} + \sum_{k=1}^{m} \bar{\Omega}_k(t) W_{1,n,k}, \tag{9}$$

$$w_2(x_n + th) = \sum_{i=0}^{r-1} \Omega_i(t) u_{2,n-i} + \sum_{k=1}^{m} \bar{\Omega}_k(t) W_{2,n,k}, \tag{10}$$

$$w_3(x_n + th) = \sum_{i=0}^{r-1} \Omega_i(t) u_{3,n-i} + \sum_{k=1}^{m} \bar{\Omega}_k(t) W_{3,n,k}, \tag{11}$$

where

$$\begin{aligned} W_{1,n,k} &:= w_1(x_{n,k}), \quad u_{1,n-i} := w_1(x_{n-i}), \\ W_{2,n,k} &:= w_2(x_{n,k}), \quad u_{2,n-i} := w_2(x_{n-i}), \\ W_{3,n,k} &:= w_3(x_{n,k}), \quad u_{3,n-i} := w_3(x_{n-i}), \end{aligned}$$

and

$$\Omega_i(t) = \Big(\prod_{j=1}^{m} \frac{t - s_j}{-i - s_j}\Big)\Big(\prod_{j=0, j\neq i}^{r-1} \frac{t+j}{-i+j}\Big), \tag{12}$$

$$\bar{\Omega}_k(t) = \Big(\prod_{i=0}^{r-1} \frac{t+i}{s_k + i}\Big)\Big(\prod_{i=1, i\neq k}^{m} \frac{t - s_i}{s_k - s_i}\Big). \tag{13}$$

We can get to the starting values $u_{1,1}, \cdots, u_{1,r-1}, u_{2,1}, \cdots, u_{2,r-1}$ and $u_{3,1}, \cdots, u_{3,r-1}$ by a classical one step method. For $x = 0$ from (8), we have

$$u_{1,0} = u_1(x_0) = u_1(0) = f_1(0), \;\; u_{2,0} = u_2(x_0) = u_2(0) = f_2(0),$$

and also by differentiating with respect to x in the third equation of (8), we get

$$\begin{aligned} 0 = \bar{f}_3'(x) + H_{32}(x,x)H_{21}(x,x)u_1(x) + \int_0^x \frac{\partial H_{32}(x,x)H_{21}(x,y)}{\partial x} u_1(y)dy \\ + \bar{H}_{32}(x,x)u_2(x) + \int_0^x \frac{\partial \bar{H}_{32}(x,y)}{\partial x} u_2(y)dy. \end{aligned} \tag{14}$$

Now, inserting u_1, u_2 from the first and second equations (8) in (14) and then by differentiating we get to $u_{3,0} = u_3(x_0) = u_3(0)$. The approximations w_1, w_2, w_3 must satisfy the following system of collocation equations:

$$\begin{cases} w_1(x_{n,i}) = f_1(x_{n,i}) + \int_0^{x_{n,i}} H_{11}(x_{n,i},y)w_1(y)dy + \int_0^{x_{n,i}} H_{12}(x_{n,i},y)w_2(y)dy \\ \qquad + \int_0^{x_{n,i}} H_{13}(x_{n,i},y)w_3(y)dy, \\ w_2(x_{n,i}) = f_2(x_{n,i}) + \int_0^{x_{n,i}} H_{21}(x_{n,i},y)w_1(y)dy + \int_0^{x_{n,i}} H_{22}(x_{n,i},y)w_2(y)dy, \\ 0 = f_3(x_{n,i}) + \int_0^{x_{n,i}} H_{32}(x_{n,i},y)w_2(y)dy. \end{cases} \tag{15}$$

By substituting equations (9)-(11) into (15), we obtain the following linear system for the unknowns $\{W_{l,n,i}\}_{l=1}^{3}$, $n = r-1, \cdots, N-1, i = 1, \cdots, m$

$$
\left\{
\begin{array}{l}
W_{1,n,i} = f_1(x_{n,i}) + h\sum_{p=0}^{r-2}\int_0^1 \Big(H_{11}(x_{n,i}, x_p+th)w_1(x_p+th) \\
+H_{12}(x_{n,i}, x_p+th)w_2(x_p+th) + H_{13}(x_{n,i}, x_p+th)w_3(x_p+th)\Big)dt \\
+h\sum_{p=r-1}^{n-1}\int_0^1 H_{11}(x_{n,i}, x_p+th)\Big(\sum_{l=0}^{r-1}\Omega_l(t)u_{1,p-l} + \sum_{k=1}^{m}\bar{\Omega}_k(t)W_{1,p,k}\Big)dt \\
+h\sum_{p=r-1}^{n-1}\int_0^1 H_{12}(x_{n,i}, x_p+th)\Big(\sum_{l=0}^{r-1}\Omega_l(t)u_{2,p-l} + \sum_{k=1}^{m}\bar{\Omega}_k(t)W_{2,p,k}\Big)dt \\
+h\sum_{p=r-1}^{n-1}\int_0^1 H_{13}(x_{n,i}, x_p+th)\Big(\sum_{l=0}^{r-1}\Omega_l(t)u_{3,p-l} + \sum_{k=1}^{m}\bar{\Omega}_k(t)W_{3,p,k}\Big)dt \\
+h\int_0^{s_i} H_{11}(x_{n,i}, x_n+th)\Big(\sum_{l=0}^{r-1}\Omega_l(t)u_{1,n-l} + \sum_{k=1}^{m}\bar{\Omega}_k(t)W_{1,n,k}\Big)dt \\
+h\int_0^{s_i} H_{12}(x_{n,i}, x_n+th)\Big(\sum_{l=0}^{r-1}\Omega_l(t)u_{2,n-l} + \sum_{k=1}^{m}\bar{\Omega}_k(t)W_{2,n,k}\Big)dt \\
+h\int_0^{s_i} H_{13}(x_{n,i}, x_n+th)\Big(\sum_{l=0}^{r-1}\Omega_l(t)u_{3,n-l} + \sum_{k=1}^{m}\bar{\Omega}_k(t)W_{3,n,k}\Big)dt, \\
\\
W_{2,n,i} = f_2(x_{n,i}) + h\sum_{p=0}^{r-2}\int_0^1 \Big(H_{21}(x_{n,i}, x_p+th)w_1(x_p+th) \\
+H_{22}(x_{n,i}, x_p+th)w_2(x_p+th)\Big)dt \\
+h\sum_{p=r-1}^{n-1}\int_0^1 H_{21}(x_{n,i}, x_p+th)\Big(\sum_{l=0}^{r-1}\Omega_l(t)u_{1,p-l} + \sum_{k=1}^{m}\bar{\Omega}_k(t)W_{1,p,k}\Big)dt \\
+h\sum_{p=r-1}^{n-1}\int_0^1 H_{22}(x_{n,i}, x_p+th)\Big(\sum_{l=0}^{r-1}\Omega_l(t)u_{2,p-l} + \sum_{k=1}^{m}\bar{\Omega}_k(t)W_{2,p,k}\Big)dt \\
+h\int_0^{s_i} H_{21}(x_{n,i}, x_n+th)\Big(\sum_{l=0}^{r-1}\Omega_l(t)u_{1,n-l} + \sum_{k=1}^{m}\bar{\Omega}_k(t)W_{1,n,k}\Big)dt \\
+h\int_0^{s_i} H_{22}(x_{n,i}, x_n+th)\Big(\sum_{l=0}^{r-1}\Omega_l(t)u_{2,n-l} + \sum_{k=1}^{m}\bar{\Omega}_k(t)W_{2,n,k}\Big)dt, \\
\\
0 = f_3(x_{n,i}) + h\sum_{p=0}^{r-2}\int_0^1 H_{32}(x_{n,i}, x_p+th)w_2(x_p+th)dt \\
+h\sum_{p=r-1}^{n-1}\int_0^1 H_{32}(x_{n,i}, x_p+th)\Big(\sum_{l=0}^{r-1}\Omega_l(t)u_{2,p-l} + \sum_{k=1}^{m}\bar{\Omega}_k(t)W_{2,p,k}\Big)dt \\
+h\int_0^{s_i} H_{32}(x_{n,i}, x_n+th)\Big(\sum_{l=0}^{r-1}\Omega_l(t)u_{2,n-l} + \sum_{k=1}^{m}\bar{\Omega}_k(t)W_{2,n,k}\Big)dt.
\end{array}
\right. \tag{16}
$$

Now, we consider (16) in matrix notation as:

$$\mathbf{A}\mathbf{W}_n = \mathbf{F}_n + \sum_{p=r-1}^{n-1} \mathbf{B}\mathbf{W}_p + \sum_{p=r-1}^{n} \mathbf{C}\mathbf{U}_p + h\sum_{p=0}^{r-2} \mathbf{V}_p, \tag{17}$$

where

$$\mathbf{W}_n = \begin{pmatrix} W_{1n} \\ W_{2n} \\ W_{3n} \end{pmatrix}, \quad \mathbf{F}_n = \begin{pmatrix} F_{1n} \\ F_{2n} \\ F_{3n} \end{pmatrix}, \quad \mathbf{U}_p = \begin{pmatrix} U_{1p} \\ U_{2p} \\ U_{3p} \end{pmatrix},$$

and

$$\mathbf{V}_p = \begin{pmatrix} V_{11}^{n,p} + V_{12}^{n,p} + V_{13}^{n,p} \\ V_{21}^{n,p} + V_{22}^{n,p} \\ V_{32}^{n,p} \end{pmatrix},$$

such that for $l = 1, 2, 3$, $W_{ln} = (W_{ln1}, \dots, W_{lnm})^T$, $F_{ln} = (f_l(x_{n,1}), \dots, f_l(x_{n,m}))^T$ and $U_{lp} = (u_{l,p-r+1}, u_{l,p-r+2}, \dots, u_{l,p})^T$. Also

$$\mathbf{A} = \begin{pmatrix} I_m - hC_{n,n}^{(1,1)} & -hC_{n,n}^{(1,2)} & -hC_{n,n}^{(1,3)} \\ -hC_{n,n}^{(2,1)} & I_m - hC_{n,n}^{(2,2)} & 0 \\ 0 & -hC_{n,n}^{(3,2)} & 0 \end{pmatrix},$$

$$\mathbf{B} = \begin{pmatrix} hC_{n,p}^{(1,1)} & hC_{n,p}^{(1,2)} & hC_{n,p}^{(1,3)} \\ hC_{n,p}^{(2,1)} & hC_{n,p}^{(2,2)} & 0 \\ 0 & hC_{n,p}^{(3,2)} & 0 \end{pmatrix},$$

$$\mathbf{C} = \begin{pmatrix} h\bar{C}_{n,p}^{(1,1)} & h\bar{C}_{n,p}^{(1,2)} & h\bar{C}_{n,p}^{(1,3)} \\ h\bar{C}_{n,p}^{(2,1)} & h\bar{C}_{n,p}^{(2,2)} & 0 \\ 0 & h\bar{C}_{n,p}^{(3,2)} & 0 \end{pmatrix},$$

such that I_m is $m \times m$ identity matrix and for $(j, q = 1, 2, 3)$ we have

$$C_{n,p}^{(j,q)} = \begin{cases} \begin{pmatrix} \displaystyle\int_0^1 H_{jq}(x_{ni}, x_p + th)\bar{\Omega}_k(t)dt \\ i, k = 1, \cdots, m \end{pmatrix}, & p = 0, \cdots, n-1, \\[2ex] \begin{pmatrix} \displaystyle\int_0^{s_i} H_{jq}(x_{ni}, x_n + th)\bar{\Omega}_k(t)dt \\ i, k = 1, \cdots, m \end{pmatrix}, & p = n, \end{cases}$$

$$\bar{C}_{n,p}^{(j,q)} = \begin{cases} \begin{pmatrix} \displaystyle\int_0^1 H_{jq}(x_{ni}, x_p + th)\Omega_l(t)dt \\ i = 1, \cdots, m, \ l = 0, \cdots, r-1, \end{pmatrix}, & p = 0, \cdots, n-1, \\[2ex] \begin{pmatrix} \displaystyle\int_0^{s_i} H_{jq}(x_{ni}, x_n + th)\Omega_l(t)dt \\ i = 1, \cdots, m, \ l = 0, \cdots, r-1, \end{pmatrix}, & p = n, \end{cases}$$

$$V_{lq}^{n,p} = \left(\int_0^1 H_{ql}(x_{n,1}, x_p + th) w_q(x_p + th)dt, \dots, \int_0^1 H_{lq}(x_{n,m}, x_p + th) w_q(x_p + th)dt \right)^T.$$

By Lemma 2.1 [26] and Theorem 1, the condition

$$|(H_{32}(x,x)H_{21}(x,x))H_{13}(x,x))| > 0, \qquad \forall x \in J,$$

ensures the invertibility of matrix $\mathbf{A}$ in (17). Consequently, we establish the following theorem guaranteeing the existence and uniqueness of the multi-step collocation solution.

Theorem 2.2. *solution $\mathbf{W}_n$. Consequently, the collocation equation (15) on each interval $\epsilon_n = [x_n, x_{n+1}](n \geq r-1)$ uniquely determines a multi-step collocation solution, expressed through the representations given in (9),(10) and (11).*

Algorithm
1. Input the values: X, N, r, m, and s_i for every i from 1 to m.
2. For $l = 1, 2, 3$, provide $u_{l,0}, \ldots, u_{l,r-1}$ and $w_l(x_p + th), p = 0, \ldots, r-2$ as the starting values.
3. Using Equations (12),(13), calculate the values for $\Omega_i(t)$ and $\bar{\Omega}_k(t)$.
4. For $r-1 \leq m \leq M-1$, solve the system of equations given by (17).
5. For the solution obtained in the previous step, take the values $\mathbf{W}_n$.
6. For $l = 1, 2, 3$ and $r-1 \leq n \leq M-1$, calculate the value $w_l(x_n + th)$ using the results from the previous steps.

2.3 The fully discretised collocation equations

To derive the discretized version of the linear system (16), we need to numerical integration. For this purpose, we can use the following two general methods:

1) Newton-Cotes formulas,

2) Gaussian Quadrature.

The Newton-Cotes formulas were derived by integrating interpolating polynomials. The error term in the interpolating polynomial of degree n involves the $(n+1)th$ derivative of the function being approximated, so a Newton-Cotes formula is exact when approximating the integral of any polynomial of degree less than or equal to n. For one step method with the order of convergence $O(h^m)$ we can use Newton-Cotes formulas so that it completely matches the approximate method. But we know that this method has disadvantages, including the following:

1. Not the most efficient. For a given number of function evaluations, other methods (like Gaussian Quadrature) can be more accurate.
2. High-order formulas can have negative weights, leading to numerical instability and round-off error.
3. Equally spaced nodes can be a limitation, as you cannot sample adaptive points where the function changes rapidly.

While for the multi-step method, the convergence order is high and a high-precision integration method is required. For this reason, we consider the Gaussian method in this section. Specifically, we employ Legendre-Gauss quadrature as:

$$\int_{-1}^{1} f(t)dt \approx \sum_{j=0}^{M} f(z_j)w_j,$$

where the Legendre Gauss quadrature points z_j are the $M+1$ roots of the Legendre polynomial $P_{M+1}(z)$ and the weights w_j are obtained by:

$$w_j = \frac{2}{(1-(z_j)^2)(P'_{M+1}(z_j))^2}, \quad j = 0, \cdots, M.$$

Legendre-Gauss quadrature is exact when approximating the integral of any polynomial of degree less than or equal to $2M+1$. This is the classical Gauss quadrature, which provides the rule of exact integration of a polynomial of maximum order. For instance, we have $(l, q = 1, 2, 3)$

$$\int_{0}^{1} H_{lq}(x_{ni}, x_p + th)\bar{\Omega}_k(t)dt \approx \frac{1}{2}\sum_{j=0}^{M} H_{lq}(x_{ni}, x_p + \frac{1}{2}(z_j+1)h)\bar{\Omega}_k(\frac{1}{2}(z_j+1))w_j,$$

$$\int_{0}^{s_i} H_{lq}(x_{ni}, x_n + th)\bar{\Omega}_k(t)dt \approx \frac{s_i}{2}\sum_{j=0}^{M} H_{lq}(x_{ni}, x_n + \frac{s_i}{2}(z_j+1)h)\bar{\Omega}_k(\frac{s_i}{2}(z_j+1))w_j.$$

The resulting approximations of u_1, u_2, u_3 are denoted by $\hat{w}_1, \hat{w}_2, \hat{w}_3$ and are defined by

$$
\begin{cases}
\hat{W}_{1,n,i} = f_1(x_{n,i}) + \frac{h}{2}\sum_{p=0}^{r-2}\sum_{j=0}^{M}\Big(H_{11}(x_{n,i}, x_p + \frac{1}{2}(z_j+1)h)w_1(x_p + \frac{1}{2}(z_j+1)h) \\
+H_{12}(x_{n,i}, x_p + \frac{1}{2}(z_j+1)h)w_2(x_p + \frac{1}{2}(z_j+1)h) + H_{13}(x_{n,i}, x_p + \frac{1}{2}(z_j+1)h)w_3(x_p + \frac{1}{2}(z_j+1)h)\Big) \\
+\frac{h}{2}\sum_{p=r-1}^{n-1}\sum_{j=0}^{M} H_{11}(x_{n,i}, x_p + \frac{1}{2}(z_j+1)h)\Big(\sum_{l=0}^{r-1}\Omega_l(\frac{1}{2}(z_j+1))u_{1,p-l} + \sum_{k=1}^{m}\bar{\Omega}_k(\frac{1}{2}(z_j+1))\hat{W}_{1,p,k}\Big) \\
+\frac{h}{2}\sum_{p=r-1}^{n-1}\sum_{j=0}^{M} H_{12}(x_{n,i}, x_p + \frac{1}{2}(z_j+1)h)\Big(\sum_{l=0}^{r-1}\Omega_l(\frac{1}{2}(z_j+1))u_{2,p-l} + \sum_{k=1}^{m}\bar{\Omega}_k(\frac{1}{2}(z_j+1))\hat{W}_{2,p,k}\Big) \\
+\frac{h}{2}\sum_{p=r-1}^{n-1}\sum_{j=0}^{M} H_{13}(x_{n,i}, x_p + \frac{1}{2}(z_j+1)h)\Big(\sum_{l=0}^{r-1}\Omega_l(\frac{1}{2}(z_j+1))u_{3,p-l} + \sum_{k=1}^{m}\bar{\Omega}_k(\frac{1}{2}(z_j+1))\hat{W}_{3,p,k}\Big) \\
+\frac{s_i h}{2}\sum_{j=0}^{M} H_{11}(x_{n,i}, x_n + \frac{s_i}{2}(z_j+1)h)\Big(\sum_{l=0}^{r-1}\Omega_l(\frac{s_i}{2}(z_j+1))u_{1,n-l} + \sum_{k=1}^{m}\bar{\Omega}_k(\frac{s_i}{2}(z_j+1))\hat{W}_{1,n,k}\Big) \\
+\frac{s_i h}{2}\sum_{j=0}^{M} H_{12}(x_{n,i}, x_n + \frac{s_i}{2}(z_j+1)h)\Big(\sum_{l=0}^{r-1}\Omega_l(\frac{s_i}{2}(z_j+1))u_{2,n-l} + \sum_{k=1}^{m}\bar{\Omega}_k(\frac{s_i}{2}(z_j+1))\hat{W}_{2,n,k}\Big) \\
+\frac{s_i h}{2}\sum_{j=0}^{M} H_{13}(x_{n,i}, x_n + \frac{s_i}{2}(z_j+1)h)\Big(\sum_{l=0}^{r-1}\Omega_l(\frac{s_i}{2}(z_j+1))u_{3,n-l} + \sum_{k=1}^{m}\bar{\Omega}_k(\frac{s_i}{2}(z_j+1))\hat{W}_{3,n,k}\Big), \\
\\
\hat{W}_{2,n,i} = f_2(x_{n,i}) + \frac{h}{2}\sum_{p=0}^{r-2}\sum_{j=0}^{M}\Big(H_{21}(x_{n,i}, x_p + \frac{1}{2}(z_j+1)h)w_1(x_p + \frac{1}{2}(z_j+1)h) \\
+H_{22}(x_{n,i}, x_p + \frac{1}{2}(z_j+1)h)w_2(x_p + \frac{1}{2}(z_j+1)h)\Big) \\
+\frac{h}{2}\sum_{p=r-1}^{n-1}\sum_{j=0}^{M} H_{21}(x_{n,i}, x_p + \frac{1}{2}(z_j+1)h)\Big(\sum_{l=0}^{r-1}\Omega_l(\frac{1}{2}(z_j+1))u_{1,p-l} + \sum_{k=1}^{m}\bar{\Omega}_k(\frac{1}{2}(z_j+1))\hat{W}_{1,p,k}\Big) \\
+\frac{h}{2}\sum_{p=r-1}^{n-1}\sum_{j=0}^{M} H_{22}(x_{n,i}, x_p + \frac{1}{2}(z_j+1)h)\Big(\sum_{l=0}^{r-1}\Omega_l(\frac{1}{2}(z_j+1))u_{2,p-l} + \sum_{k=1}^{m}\bar{\Omega}_k(\frac{1}{2}(z_j+1))\hat{W}_{2,p,k}\Big) \\
+\frac{s_i h}{2}\sum_{j=0}^{M} H_{21}(x_{n,i}, x_n + \frac{s_i}{2}(z_j+1)h)\Big(\sum_{l=0}^{r-1}\Omega_l(\frac{s_i}{2}(z_j+1))u_{1,n-l} + \sum_{k=1}^{m}\bar{\Omega}_k(\frac{s_i}{2}(z_j+1))\hat{W}_{1,n,k}\Big) \\
+\frac{s_i h}{2}\sum_{j=0}^{M} H_{22}(x_{n,i}, x_n + \frac{s_i}{2}(z_j+1)h)\Big(\sum_{l=0}^{r-1}\Omega_l(\frac{s_i}{2}(z_j+1))u_{2,n-l} + \sum_{k=1}^{m}\bar{\Omega}_k(\frac{s_i}{2}(z_j+1))\hat{W}_{2,n,k}\Big), \\
\\
0 = f_3(x_{n,i}) + \frac{h}{2}\sum_{p=0}^{r-2}\sum_{j=0}^{M} H_{32}(x_{n,i}, x_p + \frac{1}{2}(z_j+1)h)w_2(x_p + \frac{1}{2}(z_j+1)h)dt \\
+\frac{h}{2}\sum_{p=r-1}^{n-1}\sum_{j=0}^{M} H_{32}(x_{n,i}, x_p + \frac{1}{2}(z_j+1)h)\Big(\sum_{l=0}^{r-1}\Omega_l(\frac{1}{2}(z_j+1))u_{2,p-l} + \sum_{k=1}^{m}\bar{\Omega}_k(\frac{1}{2}(z_j+1))\hat{W}_{2,p,k}\Big) \\
+\frac{s_i h}{2}\sum_{j=0}^{M} H_{32}(x_{n,i}, x_n + \frac{s_i}{2}(z_j+1)h)\Big(\sum_{l=0}^{r-1}\Omega_l(\frac{s_i}{2}(z_j+1)h)u_{2,n-l} + \sum_{k=1}^{m}\bar{\Omega}_k(\frac{s_i}{2}(z_j+1)h)\hat{W}_{2,n,k}\Big).
\end{cases}
\tag{18}
$$

3 Convergence analysis

3.1 One-step method

Following the analysis in [23], for the one-step collocation method on each subinterval $\epsilon_n = [x_n, x_{n+1}]$, the approximations w_1, w_2, w_3 are polynomials of degree $m-1$. These approximations admit the representation:

$$w_1(x_n + th) = \sum_{k=1}^{m} L_k(t) W_{1,n,k}, \tag{19}$$

$$w_2(x_n + th) = \sum_{k=1}^{m} L_k(t) W_{2,n,k}, \tag{20}$$

$$w_3(x_n + th) = \sum_{k=1}^{m} L_k(t) W_{3,n,k}, \tag{21}$$

where

$$L_k(t) = \prod_{j\neq k, j=1}^{m} \frac{(t - s_j)}{(s_k - s_j)}.$$

Furthermore, the following Theorem from [23], establishes the convergence properties of the one-step collocation method.

Theorem 3.1. [23] *Assume that*

(*a*) *The assumptions of Theorem 2.1 are satisfied.*

(*b*) *The functions w_1, w_2 and w_3 represent the collocation approximations to the solutions u_1, u_2 and u_3 of system (7), as defined by equations (19), (20) and (21), respectively.*

(*c*) *For $0 < s_1 < s_2 < \ldots < s_m \leq 1$, we define*

$$\lambda = (-1)^m \prod_{i=1}^{m} \frac{1 - s_i}{s_i}.$$

Convergence of the collocation approximations w_i to the exact solutions u_i of system (7) holds for all $m \geq 3$ if and only if the stability condition $|\lambda| < 1$ is satisfied. Furthermore, when this condition holds, the approximation errors admit the following bound for sufficiently small h :

$$||u_1 - w_1||_\infty = O(h^{m-1}), \qquad ||u_2 - w_2||_\infty = O(h^m), \qquad ||u_3 - w_3||_\infty = O(h^{m-2}).$$

Furthermore, when $\lambda = -1$, the collocation method exhibits a weak instability, resulting in:

$$||u_1 - w_1||_\infty = O(h^{m-2}), \qquad ||u_2 - w_2||_\infty = O(h^m), \qquad ||u_3 - w_3||_\infty = O(h^{m-4}),$$

and for $\lambda = 1$, we have

$$||u_1 - w_1||_\infty = O(h^{m-3}), \qquad ||u_2 - w_2||_\infty = O(h^{m-1}), \qquad ||u_3 - w_3||_\infty = O(h^{m-5}).$$

3.2 Multi-step method

Now, for the multi-step method, we investigate convergence analysis as:

Theorem 3.2. *Assume that w_1, w_2 and w_3 are the collocation approximations of the exact solutions u_1, u_2 and u_3 of (7) which is defined by (9), (10) and (11). Also, the order of primary error is $m + r$, and the spectral radius of the matrix*

$$\tilde{R} = \left[\begin{array}{c|c} \mathbf{0}_{r-1,1} & \mathbf{I}_{r-1} \\ \hline \Omega_{r-1}(1) & \Omega_{r-2}(1), \ldots, \Omega_0(1) \end{array}\right], \tag{22}$$

is less than 1, ($\rho(\tilde{R}) < 1$), then the errors satisfy:

$$||u_1 - w_1||_\infty = O(h^{m+r-1}), \quad ||u_2 - w_2||_\infty = O(h^{m+r}), \quad ||u_3 - w_3||_\infty = O(h^{m+r-2}). \tag{23}$$

Proof. Using Peano theorem for interpolation and Lemma 4.1 [8], the exact solution of the IAEs system (7) can be written as:

$$u_1(x_n + th) = \sum_{i=0}^{r-1} \Omega_i(t) u_1(x_{n-i}) + \sum_{k=1}^{m} \bar{\Omega}_k(t) u_1(x_{n,k}) + O(h^{m+r}), \tag{24}$$

$$u_2(x_n + th) = \sum_{i=0}^{r-1} \Omega_i(t) u_2(x_{n-i}) + \sum_{k=1}^{m} \bar{\Omega}_k(t) u_2(x_{n,k}) + O(h^{m+r}), \tag{25}$$

$$u_3(x_n + th) = \sum_{i=0}^{r-1} \Omega_i(t) u_3(x_{n-i}) + \sum_{k=1}^{m} \bar{\Omega}_k(t) u_3(x_{n,k}) + O(h^{m+r}). \tag{26}$$

Let $e_1 = u_1 - w_1, e_2 = u_2 - w_2$ and $e_3 = u_3 - w_3$, then the collocation errors have the following representations

$$e_1(x_n + th) = \sum_{i=0}^{r-1} \Omega_i(t) e_1(x_{n-i}) + \sum_{k=1}^{m} \bar{\Omega}_k(t) e_1(x_{n,k}) + O(h^{m+r}), \tag{27}$$

$$e_2(x_n + th) = \sum_{i=0}^{r-1} \Omega_i(t) e_2(x_{n-i}) + \sum_{k=1}^{m} \bar{\Omega}_k(t) e_2(x_{n,k}) + O(h^{m+r}), \tag{28}$$

$$e_3(x_n + th) = \sum_{i=0}^{r-1} \Omega_i(t) e_3(x_{n-i}) + \sum_{k=1}^{m} \bar{\Omega}_k(t) e_3(x_{n,k}) + O(h^{m+r}), \tag{29}$$

where

$$\begin{cases} e_1(x_{n,i}) = h\sum_{p=0}^{n-1}\int_0^1 H_{11}(x_{n,i},x_p+th)e_1(x_p+th)dt + h\int_0^{s_i} H_{11}(x_{n,i},x_n+th)e_1(x_n+th)dt \\ \quad h\sum_{p=0}^{n-1}\int_0^1 H_{12}(x_{n,i},x_p+th)e_2(x_p+th)dt + h\int_0^{s_i} H_{12}(x_{n,i},x_n+th)e_2(x_n+th)dt \\ \quad h\sum_{p=0}^{n-1}\int_0^1 H_{13}(x_{n,i},x_p+th)e_3(x_p+th)dt + h\int_0^{s_i} H_{13}(x_{n,i},x_n+th)e_3(x_n+th)dt, \\ e_2(x_{n,i}) = h\sum_{p=0}^{n-1}\int_0^1 H_{21}(x_{n,i},x_p+th)e_1(x_p+th)dt + h\int_0^{s_i} H_{21}(x_{n,i},x_n+th)e_1(x_n+th)dt \\ \quad h\sum_{p=0}^{n-1}\int_0^1 H_{22}(x_{n,i},x_p+th)e_2(x_p+th)dt + h\int_0^{s_i} H_{22}(x_{n,i},x_n+th)e_2(x_n+th)dt, \\ 0 = h\sum_{p=0}^{n-1}\int_0^1 H_{32}(x_{n,i},x_p+th)e_2(x_p+th)dt + h\int_0^{s_i} H_{32}(x_{n,i},x_n+th)e_2(x_n+th)dt. \end{cases} \tag{30}$$

We now rewrite the third equation of (30) with n replaced by $n-1$ and $i=m$, subtract this equation from the third equation of (30) and divide by h:

$$\begin{aligned} &\int_0^{s_i} H_{32}(x_{n,i},x_n+th)e_2(x_n+th)dt = -\int_{c_m}^1 H_{32}(x_{n-1,m},x_{n-1}+th)e_2(x_{n-1}+th)dt \\ &-\sum_{p=0}^{n-1}\int_0^1 \Big(s_i h\frac{\partial H_{32}}{\partial t}(x_n,x_p+th) + (1-s_m)h\frac{\partial H_{32}}{\partial t}(x_n,x_p+th) + O(h)\Big)e_2(x_p+th)dt. \end{aligned} \tag{31}$$

By the hypothesis on the starting error (the order of primary error is $m+r$) and inserting (27), (28) and (29) into (30), (31), we have:

$$\tilde{\mathbf{A}}\mathbf{E}_n^{(2)} = \sum_{p=r-1}^{n-1}\tilde{\mathbf{B}}\mathbf{E}_p^{(2)} + \sum_{p=r-1}^{n}\tilde{\mathbf{C}}\mathbf{E}_p^{(1)} + O(h^{m+r}), \tag{32}$$

where

$$\mathbf{E}_n^{(2)} = \begin{pmatrix} e_{1n}^{(2)} \\ e_{2n}^{(2)} \\ e_{3n}^{(2)} \end{pmatrix}, \quad \mathbf{E}_p^{(1)} = \begin{pmatrix} e_{1p}^{(1)} \\ e_{2p}^{(1)} \\ e_{3p}^{(1)} \end{pmatrix},$$

such that for $l=1,2,3$, $e_{ln}^{(2)} = (e_l(x_{n,1}),\ldots,e_l(x_{n,m}))^T$ and $e_{lp}^{(1)} = (e_{l,p-r+1},e_{l,p-r+2},\ldots,e_{l,p})^T$. Also

$$\tilde{\mathbf{A}} = \begin{pmatrix} I_m - hC_{n,n}^{(1,1)} & -hC_{n,n}^{(1,2)} & -hC_{n,n}^{(1,3)} \\ -hC_{n,n}^{(2,1)} & I_m - hC_{n,n}^{(2,2)} & 0 \\ 0 & -C_{n,n}^{(3,2)} & 0 \end{pmatrix},$$

where the matrices $C^{(.,.)}_{n,n}$ are defined in (17) and the meaning of the matrices $\tilde{\mathbf{B}}$ and $\tilde{\mathbf{C}}$ are clear. From the proof of the Theorem 2 in [23], the inverse of the matrix $\tilde{\mathbf{A}}$ can be obtained:

$$\tilde{\mathbf{A}}^{-1} = \begin{pmatrix} 0 & \tilde{a}_{12} & \tilde{a}_{13} \\ 0 & 0 & \tilde{a}_{23} \\ \tilde{a}_{31} & \tilde{a}_{32} & \tilde{a}_{33} \end{pmatrix},$$

where

$$\tilde{a}_{12} = -\frac{1}{h}(C^{(2,1)}_{n,n})^{-1},\ \tilde{a}_{23} = -(C^{(3,2)}_{n,n})^{-1},\ \tilde{a}_{31} = -\frac{1}{h}(C^{(1,3)}_{n,n})^{-1},$$
$$\tilde{a}_{13} = -\frac{1}{h}(C^{(2,1)}_{n,n})^{-1}\Big((I_m - hC^{(2,2)}_{n,n})(C^{(3,2)}_{n,n})^{-1}\Big),$$
$$\tilde{a}_{32} = \frac{1}{h}(C^{(1,3)}_{n,n})^{-1}\Big((I_m - hC^{(1,1)}_{n,n})\tilde{a}_{12}\Big),$$
$$\tilde{a}_{33} = \frac{1}{h}(C^{(1,3)}_{n,n})^{-1}\Big((I_m - hC^{(1,1)}_{n,n})\tilde{a}_{13} + hC^{(1,2)}_{n,n}(C^{(3,2)}_{n,n})^{-1}\Big).$$

Pay attention to the structure of the inverse matrix $\tilde{\mathbf{A}}^{-1}$ such that for $\tilde{a}_{23}$, h is not present in the denominator of the fraction. Multiplying it by both sides of the equation (32), considering the recursion relation (4.11) from the proof of Theorem 4.2 [8] for $\mathbf{E}^{(1)}_p$ and continuing the same process to prove the Theorem 4.2 [8], Theorem 2 [23], we can get

$$||u_1 - w_1||_\infty = O(h^{m+r-1}), \quad ||u_2 - w_2||_\infty = O(h^{m+r}), \quad ||u_3 - w_3||_\infty = O(h^{m+r-2}), \tag{33}$$

when the spectral radius of the matrix

$$\tilde{R} = \left[\begin{array}{c|c} \mathbf{0}_{r-1,1} & \mathbf{I}_{r-1} \\ \hline \Omega_{r-1}(1) & \Omega_{r-2}(1), \ldots, \Omega_0(1), \end{array}\right], \tag{34}$$

is less than 1, ($\rho(\tilde{R}) < 1$). □

4 Numerical tests

To validate the proposed method, we present three benchmark tests evaluating both accuracy and computational efficiency. All implementations were developed in Wolfram Mathematica 11.1 and executed on an ASUS laptop with the following specifications:
(ii) Memory: 8.00 GB RAM,
(iii) Timing: All reported computational times are in seconds.

For numerical implementation parameters, we select $(r, m, s_1, s_2) = (2, 2, 0.7, 1)$, $(r, m, s_1, s_2) = (2, 2, 0.8, 0.95)$ and $(r, m, s_1, s_2, s_3) = (1, 3, \frac{1}{4}, \frac{5}{6}, 1)$. The initial conditions for the numerical scheme are determined by evaluating the known analytical solutions of the system to ensure consistency with the theoretical framework. For $r = 1, m = 3$, we consider $s_1 = \frac{1}{4}, s_2 = \frac{5}{6}$ and $s_3 = 1$ (for $r = 2, m = 2$, we consider $s_1 = 0.7$ and $s_2 = 1$) such that at these points $\rho(\tilde{R}) = 0 < 1$. Also for the case $s_m < 1$ with $r = m = 2$, we consider $s_1 = 0.8, s_2 = 0.95$ such that at these points $\rho(\tilde{R}) = 0.0533 < 1$. A comparative analysis of the numerical experiments(Tables 1-9, 12-20 and 23-27) demonstrates that the proposed numerical method in the previous section achieves significantly higher accuracy than the one-step collocation approach developed in [23], particularly for smaller step sizes.

Tables 1-9, 12-20 and 23-27 document the maximum absolute errors observed at the gridpoints providing quantitative evidence of the method's accuracy.

From Theorem 3.1, we know that the convergence order for u_3 is exactly $m-2$, then in Tables 3, 6, 14, 17 and 27, for $m=2$, one-step collocation methods is not convergent. However for multi-step method from (23), the convergence order for u_3 is exactly $m+r-2$ and then for $r=m=2$, convergence is hold. In [23], the author used Gauss points (the zeros of $P_m(2s-1)$ in which P_m denotes the Legendre polynomial) as collocation points. In these points for $m=2$ we have $\lambda=1$ and weak instability. However, in these points $\rho(\tilde{R})=1.96>1$ and multi-step method does not converge. For instance, in the Example 4.1, for $N=8$ with $(r,m,s_1,s_2)=(2,2,0.7,1)$, we have

$$||u_1-w_1||_\infty = 5.40\times 10^{+3}, \quad ||u_2-w_2||_\infty = 3.69\times 10^{+1}, \quad ||u_3-w_3||_\infty = 6.23\times 10^{+5},$$

also for $N=16$,

$$||u_1-w_1||_\infty = 3.02\times 10^{+5}, \quad ||u_2-w_2||_\infty = 5.95\times 10^{+3}, \quad ||u_3-w_3||_\infty = 8.40\times 10^{+7}.$$

To assess computational efficiency, we present detailed CPU time measurements for both methods in Tables 10, 11, 21, 22 and 28, providing direct comparisons of algorithmic performance.

We plot the order of convergence in Figures 1, 2 and 3. From (23), we see that for u_1, the convergence order approach to $m+r-1=3$, for u_2 approach to $m+r=4$ and also for u_3, this order approach to $m+r-2=2$.

Example 4.1. *Consider the index-3 IAEs (7) as:*

$$\begin{cases} u_1(x) = f_1(x) + \displaystyle\int_0^x (x+y)u_1(y)dy + \int_0^x (y+1)^2 u_2(y)dy \\ \qquad\qquad + \displaystyle\int_0^x (x^2+y^2+2)u_3(y)dy, \\ u_2(x) = f_2(x) + \displaystyle\int_0^x e^{x+y}u_1(y)dy + \int_0^x (y^2+1)u_2(y)dy, \\ 0 = f_3(x) + \displaystyle\int_0^x e^{1+y}u_2(y)dy, \qquad x\in J=[0,1], \end{cases}$$

where

$$f_1(t) = 6-t+t^2-2e^t(2-t+t^2)+2t\cos t-2(1+t)\cos t-(-1+2t+t^2)\sin t,$$
$$f_2(t) = \cos t-2t\cos t-(-1+t^2)\sin t-\tfrac{1}{2}e^t(1+e^t(-\cos t+\sin t)),$$
$$f_3(t) = -\tfrac{1}{2}e(-1+e^t(\cos t+\sin t)),$$

and the exact solution is:

$$u_1(x)=\sin x, \quad u_2(x)=\cos x, \quad u_3(x)=e^x.$$

N	Collocation method [23]	Multi-step method
8	3.83×10^{-2}	1.70×10^{-4}
16	2.14×10^{-2}	4.99×10^{-5}
32	1.13×10^{-2}	1.05×10^{-5}
64	5.96×10^{-3}	1.80×10^{-6}
128	3.07×10^{-3}	2.72×10^{-7}
256	1.55×10^{-3}	3.87×10^{-8}

Table 1: $\|u_1 - w_1\|_\infty$ with $(r, m, s_1, s_2) = (2, 2, 0.7, 1)$ in Example 4.1.

N	Collocation method [23]	Multi-step method
8	7.67×10^{-4}	2.96×10^{-6}
16	1.93×10^{-4}	2.34×10^{-7}
32	4.86×10^{-5}	1.63×10^{-8}
64	1.21×10^{-5}	1.07×10^{-9}
128	3.04×10^{-6}	6.89×10^{-11}
256	7.62×10^{-7}	4.67×10^{-12}

Table 2: $\|u_2 - w_2\|_\infty$ with $(r, m, s_1, s_2) = (2, 2, 0.7, 1)$ in Example 4.1.

N	Collocation method [23]	Multi-step method
8	3.61×10^{-1}	1.10×10^{-2}
16	4.61×10^{-1}	9.47×10^{-3}
32	5.17×10^{-1}	4.93×10^{-3}
64	5.45×10^{-1}	1.89×10^{-3}
128	5.59×10^{-1}	5.99×10^{-4}
256	5.66×10^{-1}	1.74×10^{-4}

Table 3: $\|u_3 - w_3\|_\infty$ with $(r, m, s_1, s_2) = (2, 2, 0.7, 1)$ in Example 4.1.

N	Collocation method [23]	Multi-step method
8	3.73×10^{-2}	1.23×10^{-4}
16	2.09×10^{-2}	3.33×10^{-5}
32	1.10×10^{-2}	6.64×10^{-6}
64	5.80×10^{-3}	1.09×10^{-6}
128	5.80×10^{-3}	1.60×10^{-7}
256	1.51×10^{-3}	2.43×10^{-8}

Table 4: $\|u_1 - w_1\|_\infty$ with $(r, m, s_1, s_2) = (2, 2, 0.8, 0.95)$ in Example 4.1.

N	Collocation method [23]	Multi-step method
8	6.67×10^{-4}	2.38×10^{-6}
16	1.69×10^{-4}	1.80×10^{-7}
32	4.26×10^{-5}	1.23×10^{-8}
64	1.06×10^{-5}	8.00×10^{-10}
128	2.67×10^{-6}	5.10×10^{-11}
256	6.69×10^{-7}	4.03×10^{-12}

Table 5: $\|u_2 - w_2\|_\infty$ with $(r, m, s_1, s_2) = (2, 2, 0.8, 0.95)$ in Example 4.1.

N	Collocation method [23]	Multi-step method
8	3.20×10^{-1}	6.50×10^{-3}
16	4.15×10^{-1}	4.96×10^{-3}
32	4.67×10^{-1}	4.37×10^{-3}
64	4.93×10^{-1}	8.52×10^{-4}
128	5.06×10^{-1}	2.60×10^{-4}
256	5.15×10^{-1}	7.07×10^{-5}

Table 6: $\|u_3 - w_3\|_\infty$ with $(r, m, s_1, s_2) = (2, 2, 0.8, 0.95)$ in Example 4.1.

N	Collocation method [23]	Multi-step method
8	1.92×10^{-4}	2.07×10^{-5}
16	5.11×10^{-5}	3.32×10^{-6}
32	1.31×10^{-5}	4.69×10^{-7}
64	3.44×10^{-6}	6.23×10^{-8}
128	8.41×10^{-7}	8.57×10^{-9}
256	2.11×10^{-7}	6.17×10^{-9}

Table 7: $\|u_1 - w_1\|_\infty$ with $(r, m, s_1, s_2, s_3) = (1, 3, \frac{1}{4}, \frac{5}{6}, 1)$ in Example 4.1.

N	Collocation method [23]	Multi-step method
8	8.18×10^{-6}	2.46×10^{-7}
16	1.04×10^{-6}	1.57×10^{-8}
32	1.32×10^{-7}	9.89×10^{-10}
64	1.66×10^{-8}	6.19×10^{-11}
128	2.04×10^{-9}	4.12×10^{-12}
256	2.60×10^{-10}	1.32×10^{-12}

Table 8: $\|u_2 - w_2\|_\infty$ with $(r, m, s_1, s_2, s_3) = (1, 3, \frac{1}{4}, \frac{5}{6}, 1)$ in Example 4.1.

N	Collocation method [23]	Multi-step method
8	2.47×10^{-3}	2.13×10^{-3}
16	1.36×10^{-3}	8.69×10^{-4}
32	7.42×10^{-4}	2.69×10^{-4}
64	3.71×10^{-4}	7.43×10^{-5}
128	1.88×10^{-4}	4.12×10^{-5}
256	9.50×10^{-5}	1.22×10^{-5}

Table 9: $\|u_3 - w_3\|_\infty$ with $(r, m, s_1, s_2, s_3) = (1, 3, \frac{1}{4}, \frac{5}{6}, 1)$ in Example 4.1.

N	Collocation method [23]	Multi-step method
8	1.54	1.96
16	2.40	3.00
32	8.43	8.83
64	31.52	32.63
128	143.82	153.57
256	565.32	575.50

Table 10: CPU time(sec) with $(r, m, s_1, s_2) = (2, 2, 0.7, 1)$ in Example 4.1.

N	Collocation method [23]	Multi-step method
8	1.64	1.95
16	3.45	4.01
32	13.01	14.83
64	50.64	52.46
128	207.07	224.51
256	877.95	934.51

Table 11: CPU time(sec) with $(r, m, s_1, s_2, s_3) = (1, 3, \frac{1}{4}, \frac{5}{6}, 1)$ in Example 4.1.

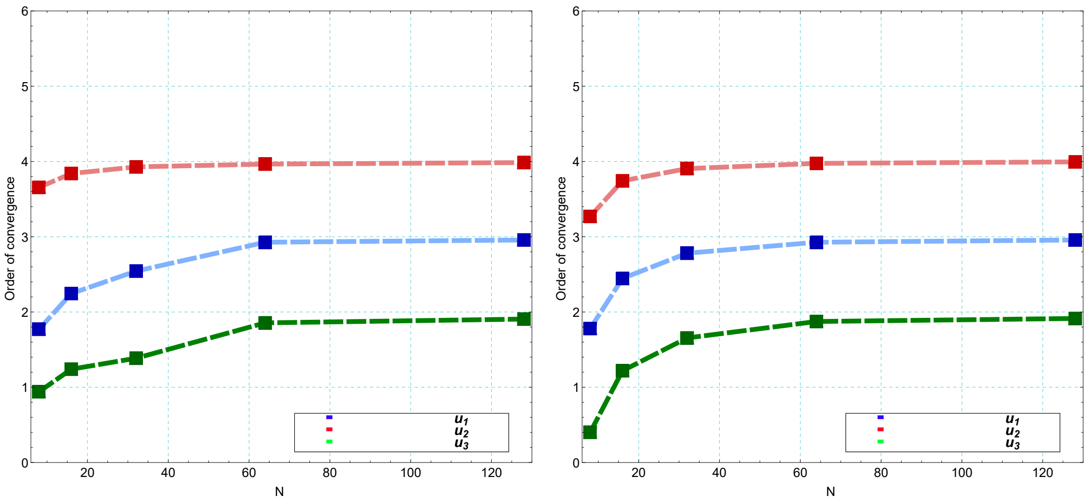

Figure 1: The convergence order for multi-step method with $(r, m, s_1, s_2, s_3) = (2, 2, 0.7, 1)$ in Example 4.1 (left). The convergence order for multi-step method with $(r, m, s_1, s_2, s_3) = (2, 2, 0.7, 1)$ in Example 4.2 (right).

Example 4.2. *Consider the index-3 IAEs (7) as:*

$$\begin{cases} u_1(x) = f_1(x) + \int_0^x e^{(2x-y)} u_1(y)dy + \int_0^x (x^2+4)u_2(y)dy \\ \qquad + \int_0^x (x+y^2+4)u_3(y)dy, \\ u_2(x) = f_2(x) + \int_0^x e^{x^2+1} u_1(y)dy + \int_0^x (x+1)u_2(y)dy, \\ 0 = f_3(x) + \int_0^x (x^2+1)^2 u_2(y)dy, \qquad x \in J = [0,1], \end{cases}$$

where

$$\begin{aligned} f_1(t) &= e^t t - \tfrac{1}{2}e^{(2t)}t^2 - \tfrac{1}{2}(4+t^2)\log(1+t^2) + \tfrac{1}{4}(-2t\cos 2t - (7+2t+2t^2)\sin 2t), \\ f_2(t) &= -e^{(1+t^2)}(1+e^t(-1+t)) + \tfrac{t}{(1+t^2)} - \tfrac{1}{2}(1+t)\log(1+t^2), \\ f_3(t) &= -\tfrac{1}{2}(1+t^2)^2\log(1+t^2), \end{aligned}$$

and the exact solution is:

$$u_1(x) = xe^x, \quad u_2(x) = \frac{x}{x^2+1}, \quad u_3(x) = \cos 2x.$$

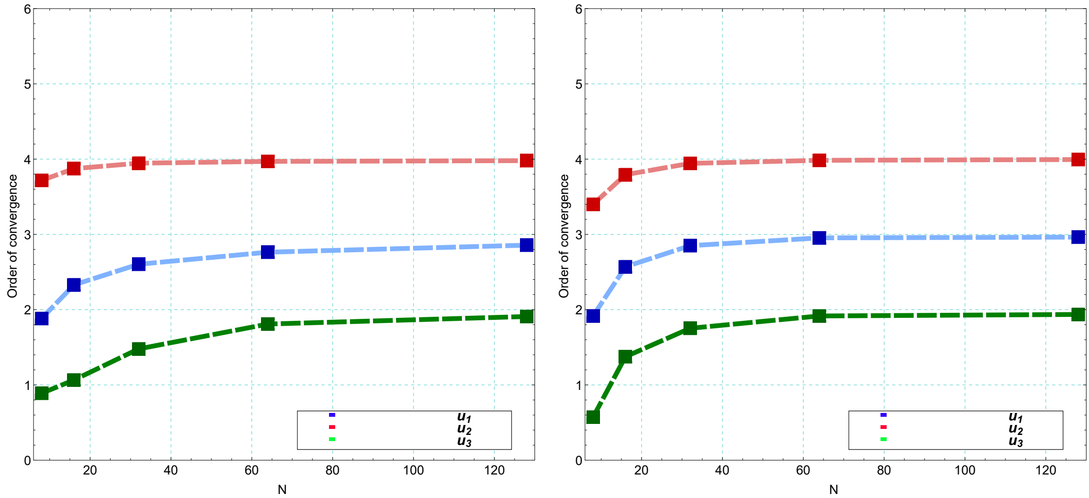

Figure 2: The convergence order for multi-step method with $(r, m, s_1, s_2, s_3) = (2, 2, 0.8, 0.95)$ in Example 4.1 (left). The convergence order for multi-step method with $(r, m, s_1, s_2, s_3) = (2, 2, 0.8, 0.95)$ in Example 4.2 (right).

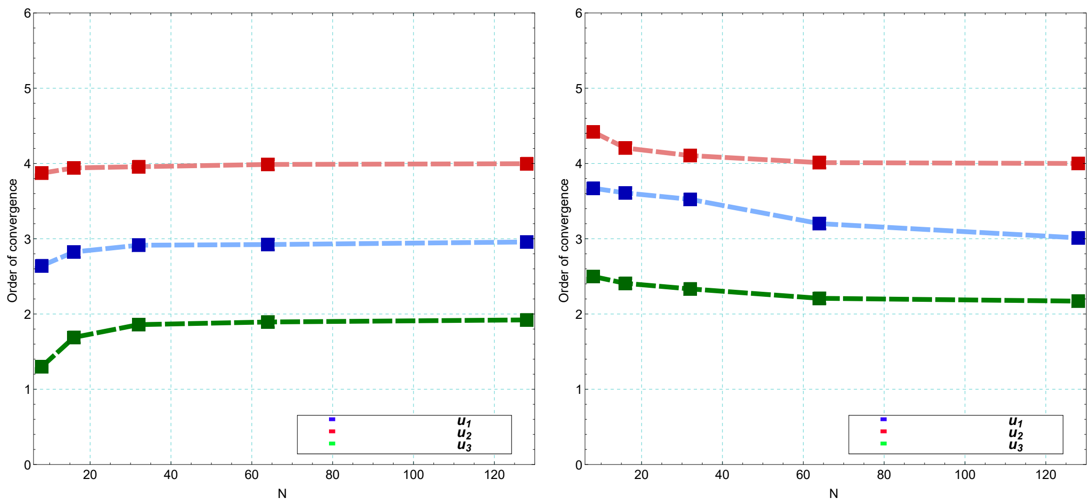

Figure 3: The convergence order for multi-step method with $(r, m, s_1, s_2, s_3) = (1, 3, \frac{1}{4}, \frac{5}{6}, 1)$ in Example 4.1 (left). The convergence order for multi-step method with $(r, m, s_1, s_2, s_3) = (1, 3, \frac{1}{4}, \frac{5}{6}, 1)$ in Example 4.2 (right).

Example 4.3. *Consider the IAEs of index-3 as:*

$$SU(x) = F(x) + \int_0^x H(x, y)U(y)dy,$$

N	Collocation method [23]	Multi-step method
8	2.00×10^{-2}	2.23×10^{-3}
16	1.09×10^{-2}	6.51×10^{-4}
32	5.65×10^{-3}	1.19×10^{-4}
64	2.88×10^{-3}	1.73×10^{-5}
128	1.45×10^{-3}	2.28×10^{-6}
256	7.29×10^{-4}	2.90×10^{-7}

Table 12: $\|u_1 - w_1\|_\infty$ with $(r, m, s_1, s_2) = (2, 2, 0.7, 1)$ in Example 4.2.

N	Collocation method [23]	Multi-step method
8	1.12×10^{-3}	4.11×10^{-5}
16	2.87×10^{-4}	4.26×10^{-6}
32	7.10×10^{-5}	3.18×10^{-7}
64	1.77×10^{-5}	2.12×10^{-8}
128	4.44×10^{-6}	1.35×10^{-9}
256	1.11×10^{-6}	8.48×10^{-11}

Table 13: $\|u_2 - w_2\|_\infty$ with $(r, m, s_1, s_2) = (2, 2, 0.7, 1)$ in Example 4.2.

N	Collocation method [23]	Multi-step method
8	6.72×10^{-2}	8.43×10^{-2}
16	9.45×10^{-2}	6.38×10^{-2}
32	1.07×10^{-1}	2.74×10^{-2}
64	1.13×10^{-1}	8.70×10^{-3}
128	1.17×10^{-1}	2.37×10^{-3}
256	1.18×10^{-1}	6.11×10^{-4}

Table 14: $\|u_3 - w_3\|_\infty$ with $(r, m, s_1, s_2) = (2, 2, 0.7, 1)$ in Example 4.2.

N	Collocation method [23]	Multi-step method
8	1.98×10^{-2}	1.62×10^{-3}
16	1.07×10^{-2}	4.31×10^{-4}
32	5.53×10^{-3}	7.26×10^{-5}
64	2.81×10^{-3}	1.00×10^{-5}
128	1.41×10^{-3}	1.30×10^{-6}
256	7.12×10^{-4}	1.64×10^{-7}

Table 15: $\|u_1 - w_1\|_\infty$ with $(r, m, s_1, s_2) = (2, 2, 0.8, 0.95)$ in Example 4.2.

N	Collocation method [23]	Multi-step method
8	9.85×10^{-4}	3.55×10^{-5}
16	2.49×10^{-4}	3.36×10^{-6}
32	6.23×10^{-5}	2.42×10^{-7}
64	1.59×10^{-5}	1.57×10^{-8}
128	3.90×10^{-6}	9.97×10^{-10}
256	9.57×10^{-7}	6.25×10^{-11}

Table 16: $\|u_2 - w_2\|_\infty$ with $(r, m, s_1, s_2) = (2, 2, 0.8, 0.95)$ in Example 4.2.

N	Collocation method [23]	Multi-step method
8	5.92×10^{-2}	4.86×10^{-2}
16	8.44×10^{-2}	3.27×10^{-2}
32	9.72×10^{-2}	1.26×10^{-2}
64	1.03×10^{-1}	3.75×10^{-3}
128	1.06×10^{-1}	9.93×10^{-4}
256	1.07×10^{-1}	2.53×10^{-4}

Table 17: $\|u_3 - w_3\|_\infty$ with $(r, m, s_1, s_2) = (2, 2, 0.8, 0.95)$ in Example 4.2.

N	Collocation method [23]	Multi-step method
8	2.58×10^{-3}	2.06×10^{-4}
16	6.87×10^{-4}	1.62×10^{-5}
32	1.74×10^{-4}	1.200×10^{-6}
64	2.38×10^{-5}	9.74×10^{-8}
128	1.09×10^{-5}	1.20×10^{-8}
256	2.74×10^{-6}	2.12×10^{-9}

Table 18: $\|u_1 - w_1\|_\infty$ with $(r, m, s_1, s_2, s_3) = (1, 3, \frac{1}{4}, \frac{5}{6}, 1)$ in Example 4.2.

N	Collocation method [23]	Multi-step method
8	5.72×10^{-5}	4.46×10^{-6}
16	7.50×10^{-6}	2.08×10^{-7}
32	9.49×10^{-7}	1.21×10^{-8}
64	1.19×10^{-7}	7.56×10^{-10}
128	1.48×10^{-8}	4.72×10^{-11}
256	1.84×10^{-9}	3.04×10^{-12}

Table 19: $\|u_2 - w_2\|_\infty$ with $(r, m, s_1, s_2, s_3) = (1, 3, \frac{1}{4}, \frac{5}{6}, 1)$ in Example 4.2.

N	Collocation method [23]	Multi-step method
8	2.53×10^{-2}	1.04×10^{-2}
16	1.40×10^{-2}	1.85×10^{-3}
32	7.30×10^{-3}	2.68×10^{-4}
64	2.70×10^{-3}	3.52×10^{-5}
128	1.86×10^{-3}	5.52×10^{-6}
256	9.35×10^{-4}	3.81×10^{-6}

Table 20: $\|u_3 - w_3\|_\infty$ with $(r, m, s_1, s_2, s_3) = (1, 3, \frac{1}{4}, \frac{5}{6}, 1)$ in Example 4.2.

N	Collocation method [23]	Multi-step method
8	1.48	1.83
16	3.09	3.25
32	8.24	8.89
64	31.73	32.00
128	138.47	143.51
256	548.39	589.86

Table 21: CPU time(sec) with $(r, m, s_1, s_2) = (2, 2, 0.7, 1)$ in Example 4.2.

N	Collocation method [23]	Multi-step method
8	1.79	1.85
16	3.93	4.11
32	12.70	12.71
64	46.46	47.46
128	187.96	194.84
256	739.21	761.51

Table 22: CPU time(sec) with $(r, m, s_1, s_2, s_3) = (1, 3, \frac{1}{4}, \frac{5}{6}, 1)$ in Example 4.2.

where $S = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$, $H(x,y) = \begin{pmatrix} H_{11}(x,y) & H_{12}(x,y) & H_{13}(x,y) \\ H_{21}(x,y) & H_{22}(x,y) & \mathbf{0}_{2\times 1} \\ \mathbf{0}_{1\times 2} & H_{32}(x,y) & 0 \end{pmatrix}$, *such that*

$$H_{11}(x,y) = \begin{pmatrix} e^{2x-y} & x^2+y \\ e^{x+y} & y^2+3x \end{pmatrix}, \qquad H_{12}(x,y) = \begin{pmatrix} x^2+4 & \sin(x+y) \\ \cos(x+1) & y^2+x^2+2 \end{pmatrix},$$

$$H_{13}(x,y) = \begin{pmatrix} x+y^2+4 \\ x^2+4 \end{pmatrix}, \qquad H_{21}(x,y) = \begin{pmatrix} e^{x^2+1} & x^4+y^2+1 \\ (x+4)^2+1 & (y+x+1)^2 \end{pmatrix},$$

$$H_{22}(x,y) = \begin{pmatrix} x+1 & y+x^4+2 \\ e^x+4 & y^4+x+2 \end{pmatrix}, \qquad H_{32}(x,y) = \begin{pmatrix} (x^2+1)^2, & y^6+x^4+4 \end{pmatrix},$$

$$U(x) = \begin{pmatrix} u_1(x), u_2(x), u_3(x), u_4(x), u_5(x) \end{pmatrix}^T, \qquad F(t) = \begin{pmatrix} f_1(x), f_2(x), f_3(x), f_4(x), f_5(x) \end{pmatrix}^T,$$

and F such that the exact solution is:

$$u_1(x) = e^{3x},\ u_2(x) = \cos x,\ u_3(x) = \frac{1}{x^2+1},\ u_4(x) = \sinh x,\ u_5(t) = \cosh(2x).$$

N	Collocation method [23]	Multi-step method
8	1.09×10^{-1}	2.55×10^{-3}
16	2.90×10^{-2}	2.13×10^{-4}
32	7.67×10^{-3}	1.56×10^{-5}
64	2.05×10^{-3}	1.06×10^{-6}
128	5.74×10^{-4}	1.02×10^{-7}
256	1.73×10^{-4}	6.59×10^{-8}

Table 23: $\|u_1 - w_1\|_\infty$ with $(r, m, s_1, s_2) = (2, 2, 0.7, 1)$ in Example 4.3.

N	Collocation method [23]	Multi-step method
8	6.76×10^{-2}	2.20×10^{-3}
16	1.83×10^{-2}	2.00×10^{-4}
32	4.95×10^{-3}	1.53×10^{-5}
64	1.36×10^{-3}	1.04×10^{-6}
128	3.95×10^{-4}	9.99×10^{-8}
256	1.25×10^{-4}	5.88×10^{-8}

Table 24: $\|u_2 - w_2\|_\infty$ with $(r, m, s_1, s_2) = (2, 2, 0.7, 1)$ in Example 4.3.

N	Collocation method [23]	Multi-step method
8	1.06×10^{-2}	2.46×10^{-5}
16	2.83×10^{-3}	1.26×10^{-6}
32	7.23×10^{-4}	4.46×10^{-8}
64	1.82×10^{-4}	8.87×10^{-10}
128	4.57×10^{-5}	3.21×10^{-10}
256	1.14×10^{-5}	9.89×10^{-11}

Table 25: $\|u_3 - w_3\|_\infty$ with $(r, m, s_1, s_2) = (2, 2, 0.7, 1)$ in Example 4.3.

N	Collocation method [23]	Multi-step method
8	1.44×10^{-2}	1.48×10^{-5}
16	3.65×10^{-3}	6.91×10^{-7}
32	9.23×10^{-4}	6.85×10^{-8}
64	2.24×10^{-4}	5.39×10^{-9}
128	5.58×10^{-5}	5.75×10^{-10}
256	1.38×10^{-5}	3.99×10^{-10}

Table 26: $\|u_4 - w_4\|_\infty$ with $(r, m, s_1, s_2) = (2, 2, 0.7, 1)$ in Example 4.3.

N	Collocation method [23]	Multi-step method
8	1.04×10^{0}	7.62×10^{-2}
16	5.57×10^{-1}	1.69×10^{-2}
32	6.91×10^{-1}	2.77×10^{-3}
64	7.51×10^{-1}	4.10×10^{-4}
128	7.99×10^{-1}	1.09×10^{-4}
256	8.37×10^{-1}	9.80×10^{-5}

Table 27: $\|u_5 - w_5\|_\infty$ with $(r, m, s_1, s_2) = (2, 2, 0.7, 1)$ in Example 4.3.

N	Collocation method [23]	Multi-step method
8	1.93	2.56
16	7.43	8.00
32	27.06	29.13
64	133.29	142.63
128	559.90	565.52
256	1773.80	1779.20

Table 28: CPU time(sec) with $(r, m, s_1, s_2) = (2, 2, 0.7, 1)$ in Example 4.3.

5 Conclusion

Our analysis demonstrates that the proposed multi-step numerical method constitutes a robust and accurate computational framework to solve index-3 IAEs. The numerical experiments conducted on representative test problems confirm strong alignment between theoretical predictions and practical implementations, validating the method's reliability. Benchmark tests (Examples 4.1, 4.2 and 4.3) verify:

i Expected convergence rates,

ii Consistency with theoretical error bounds.

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Conflicts of Interest The authors declare no conflict of interest.

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